Toric non-abelian Hodge theory II

joint with Nick Proudfoot

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Non-abelian Hodge theory

- Simpson (1990), Hitchin (1987) for Riemann surfaces
- G complex reductive algebraic group, e.g. $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$
  C smooth complex projective curve (w. decorations)
- $\mathcal{M}_{\mathrm{B}} := H^1_{\mathrm{B}}(C, \mathrm{G}) = \left\{ \begin{array}{c} \text{moduli of representations} \\ \text{of } \pi_1(C) \to \mathrm{G} \end{array} \right\}$
- $\mathcal{M}_{\mathrm{DR}} := H^1_{\mathrm{DR}}(C, \mathrm{G}) = \{\text{moduli of flat G-connections on } C\}$
- $\mathcal{M}_{\mathrm{Dol}} := H^1_{\mathrm{Dol}}(C, \mathrm{G}) = \{\text{moduli of G-Higgs bundles on } C\}$
- Non-Abelian Hodge Theorem: $\mathcal{M}_{\mathrm{Dol}} \cong_{diff} \mathcal{M}_{\mathrm{DR}} \overset{\tau_{RH}}{\cong}_{an} \mathcal{M}_{\mathrm{B}}$
- Hitchin map: $\begin{array}{ccc} \chi : \mathcal{M}_{\mathrm{Dol}} & \to & \mathcal{A} \\ (E, \phi) & \mapsto & CharPol(\phi) \end{array}$
  proper, integrable system
- often $0 \in \mathcal{A}$ when $\chi^{-1}(0) \sim \mathcal{M}_{\mathrm{Dol}}$ nilpotent cone
- $C \cong \mathbb{P}^1 \rightsquigarrow \mathcal{M}^*_{\mathrm{DR}} := \{\text{moduli of flat connections on } C \times \mathrm{G}\}$
- $\mathcal{M}^*_{\mathrm{DR}} \subset \mathcal{M}_{\mathrm{DR}}$ open, $\mathcal{M}^*_{\mathrm{DR}} \cong Q$ star-shaped quiver variety

Conjectures on MHS on $H^*(\mathcal{M}_{\mathrm{B}})$

Conjecture

1. *(Hodge-Tate)*
$$h^{p,q}(H^*(\mathcal{M}_B)) \neq 0 \Rightarrow p = q$$
2. *(Curious Hard Lefschetz)*
$\alpha := [\Re(\Omega)] \in H^{2;2,2}(\mathcal{M}_{\mathrm{B}})$
$$\begin{array}{cccc} L^l : & Gr^W_{\dim -2l} H^{i-l}(\mathcal{M}) & \overset{\cong}{\to} & Gr^W_{\dim +2l} H^{i+l}(\mathcal{M}) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$
3. *(purity conjecture)*
$$W_k H^k(\mathcal{M}_{\mathrm{B}}) \overset{\tau^*_{RH}}{\cong} H^k(\mathcal{M}^*_{\mathrm{DR}})$$
4. *($P = W$)*
perverse filtration P on $H^*(\mathcal{M}_{\mathrm{Dol}})$ induced by Hitchin map χ
$$W_{2k} H^*(\mathcal{M}_{\mathrm{B}}) = P_k H^*(\mathcal{M}_{\mathrm{Dol}})$$

- proved for $\mathrm{G} = \mathrm{GL}_2$ and many consintency checks

Toric hyperkähler varieties

- Bielawski–Dancer (2000) Hausel–Sturmfels (2002)
- $A \in M_{dxn}(\mathbb{Z})$ surj. $\rightsquigarrow 0 \to \mathbb{Z}^{n-d} \overset{B}{\to} \mathbb{Z}^n \overset{A}{\to} \mathbb{Z}^d \to 0$
- taking *Hom* to $\mathbb{T} := \mathbb{C}^\times \rightsquigarrow 0 \leftarrow \mathbb{T}^{n-d} \overset{B^T}{\leftarrow} \mathbb{T}^n \overset{A^T}{\leftarrow} \mathbb{T}^d \to 0$
- $\mathbb{T}^d \subset \mathbb{T}^n \circlearrowright T^*\mathbb{C}^n$; moment map
$$\begin{array}{cccc} \nu_A : & (T^*\mathbb{C})^n & \to & (\mathfrak{t}^d)^* \\ & {\scriptstyle (x_i y_i)_i} \downarrow & & \wr\Vert \\ & \mathbb{C}^n & \overset{A}{\to} & \mathbb{C}^d \end{array}$$
- $Q_A^\xi := \nu_A^{-1}(\xi)//\mathbb{T}^d$ *toric hyperkähler variety* of dim $= 2(n-d)$
- $\mathbb{T}^{n-d} \circlearrowright Q_A^\xi$ with moment map $\nu : Q_A^\xi \to (\mathfrak{t}^{n-d})^*$ whose discriminental locus is a hyperplane arrangement $\mathcal{H}_A \subset (\mathfrak{t}^{n-d})^*$ modeled on $B^T = [b_1, \ldots, b_n] \in \mathbb{Z}^{n-d}$
- $H^*(Q_A^\xi)$ understood from the combinatorics of $\mathcal{H}_A$
  e.g. $\dim H^*(Q_A^\xi) = \#$ vertices of $\mathcal{H}_A$
- example: for any quiver Γ with n edges and $d+1$ vertices
  $\rightsquigarrow A_\Gamma(e_{ij}) = v_i - v_j$ a surjective matrix $A_\Gamma \in M_{d\times n}(\mathbb{Z})$
  $\rightsquigarrow Q_\Gamma^\xi := Q_{A_\Gamma}^\xi$ *toric quiver variety*

Toric character varieties

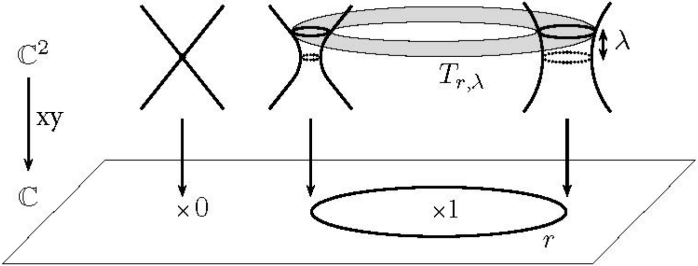
- Crawley-Boevey–Shaw (2006)
- $Z = \mathbb{C}^2 \setminus \{xy - 1 = 0\}$ with symplectic form $\Omega = \frac{dx \wedge dy}{xy-1}$ and usual $\mathbb{T}$-action is quasi-Hamiltonian with moment map
$$\begin{array}{cccc} \mu : & Z & \to & \mathbb{T} \\ & (x,y) & \mapsto & xy-1 \end{array}$$
- $\mathbb{T}^d \subset \mathbb{T}^n \circlearrowright Z^n$ with moment map $\begin{array}{cccc} \mu_A : & Z^n & \to & \mathbb{T}^d \\ & \mu^n \downarrow & & \wr\!\| \\ & \mathbb{T}^n & \stackrel{A}{\to} & \mathbb{T}^d \end{array}$
- for generic $\zeta \in \mathbb{T}^d$ define $\mathcal{M}_{\mathrm{B}}^{\zeta} := \mu_A^{-1}(\zeta) // \mathbb{T}^d$ *toric Betti space* of dim $= 2(n-d)$ with symplectic form $\Omega \in \Omega^2(\mathcal{M}_{\mathrm{B}}^{\zeta})$ of Alexeev-Malkin-Meinrenken (1998)
- Γ quiver $\rightsquigarrow A = A_\Gamma \rightsquigarrow \mathcal{M}_{\mathrm{B}}^{\zeta}$ multiplicative quiver variety of Crawley-Boevey–Shaw (2006)

Special Lagrangian fibration on $\mathcal{M}_{\mathrm{B}}^{\zeta}$

- Auroux (2009): $\chi : \begin{array}{ccc} Z & \to & \mathbb{R}^2 \\ (x,y) & & (log(|xy-1|), |x|^2 - |y|^2) \end{array}$

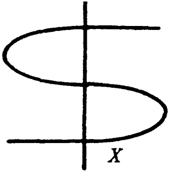
  proper special Lagrangian fibration:

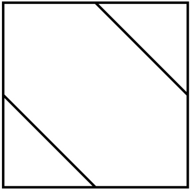
  $$\chi^{-1}(r,\lambda) = T_{r,\lambda} \cong \begin{cases} \mathbb{T}^2_{\mathbb{R}} \cong U(1)^2 & (r,\lambda) \neq (0,0) \\ \text{pinched torus} & (r,\lambda) = (0,0) \end{cases}$$



- $\rightsquigarrow \chi_A : \mathcal{M}_{\mathrm{B}}^{\zeta} \to (\mathbb{R}^2)^{n-d}$ proper special Lagrangian fibration; "toric Hitchin map in the Betti complex structure"
  degeneracy locus of χ_A is hyperplane arrangement $\mathcal{H}_A$ in $(\mathbb{R}^2)^{n-d}$ modelled on vector configuration $[b_1, \ldots, b_n] \in \mathbb{Z}^{n-d}$

Toroidal core

- $\zeta \in \mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^d_{\mathbb{C}} \rightsquigarrow \mathcal{H}_A$ linear hyperplane arrangement $\rightsquigarrow \mathcal{C}^{\zeta}_A := \chi_A^{-1}(0)$ *toroidal core*: non-normal compact toric variety over a toroidal hyperplane arrangement
- Γ quiver $\rightsquigarrow \mathcal{C}^{\zeta}_{A_\Gamma} \cong_{diff} \overline{Jac}_{\zeta}(C_\Gamma)$ compactified Jacobian of reducible nodal rational curve C_Γ of Oda-Seshadri (1979)
- e.g. $C_\Gamma \cong$ 

$\overline{Jac}_{\zeta}(C_\Gamma) \cong$ 

Theorem (Hausel-Proudfoot 2015)

$\zeta \in \mathbb{T}^d_{\mathbb{R}} \subset \mathbb{T}^d_{\mathbb{C}}$ *generic* $\rightsquigarrow \mathcal{C}^{\zeta}_A \subset \mathcal{M}^{\zeta}_{\mathrm{B}}$ *is a homotopy equivalence*

Toric Hodge-Tate and Curious Hard Lefschetz

Theorem (Hausel–Proudfoot, 2015)

$H^*(\mathcal{M}_{\mathrm{B}}^{\zeta})$ is Hodge-Tate and satisfies Curious Hard Lefschetz.

- sketch of proof:
- define $Z^x := Z \setminus \{x = 0\} \cong \mathbb{T}^2$ "toric cluster torus"
- $S \subset \{1, \ldots, n\} \rightsquigarrow (\mathcal{M}_{\mathrm{B}}^{\zeta})_S := Z^S \times (Z^x)^{S^c} ////_{\zeta}^{qH} \mathbb{T}^d \subset \mathcal{M}_{\mathrm{B}}^{\zeta}$
- $b_S \subset \{b_1, \ldots, b_n\} \subset \mathbb{Z}^{n-d}$ linearly independent $\rightsquigarrow$

  $(\mathcal{M}_{\mathrm{B}}^{\zeta})_S \cong Z^{|S|} \times \mathbb{T}^{n-d-|S|}$ in particular satisfies HT and CHL
- $(\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_1} \cap (\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_2} = (\mathcal{M}_{\mathrm{B}}^{\zeta})_{S_1 \cap S_2}$
- claim: $\mathcal{M}_{\mathrm{B}}^{\zeta} = \bigcup_{b_S \text{ lin. ind.}} (\mathcal{M}_{\mathrm{B}}^{\zeta})_S$
- result follows from Mayer-Vietoris

Toric purity

Theorem (Hausel–Proudfoot, 2015)

$$W_k H^k(\mathcal{M}_{\mathrm{B}}^{e^{\xi}}) \cong H^k(Q_A^{\xi})$$

- proof: define $\tau_{RH} : \mathbb{C}^2 \to Z$:

$$\begin{array}{ccc} (x,y) \in \mathbb{C}^2 & \xrightarrow{\tau_{RH}} & \begin{cases} \left(x, \frac{\exp(xy)+1}{x}\right) \in Z & x \neq 0 \\ (0,y) \in Z & x = 0 \end{cases} \\ xy \downarrow & & \downarrow xy - 1 \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^{\times} \end{array}$$

- $\rightsquigarrow \tau_{RH} : Q_A^{\xi} \to \mathcal{M}_{\mathrm{B}}^{e^{\xi}}$
  $\rightsquigarrow \tau_{RH}^* : W_k H^k(\mathcal{M}_{\mathrm{B}}^{e^{\xi}}) \to H^k(Q_A^{\xi})$ is surjective
- $\dim(W_* H^*(\mathcal{M}_{\mathrm{B}}^{e^{\xi}})) \overset{CHL}{=} \dim(H^{mid}(\mathcal{M}_{\mathrm{B}}^{e^{\xi}})) = \dim(H^{top}(\mathcal{C}_A^{e^{\xi}}))$
  $=$ #top dim regions in toroidal hyperplane arrangement
  $=$ #vertices of hyperplane arrangement $= \dim(H^*(Q_A^{\xi}))$ ■

Comments on toric $P = W$

- recall $\chi : Z \to \mathbb{R}^2$
- $\chi^{-1}(\Delta) \cong_{diff} T$ where $T \to \Delta$ is the Tate curve
- $\rightsquigarrow$ a neighbourhood of $\mathcal{C}_A^\zeta \subset \mathcal{M}_{\mathrm{B}}^\zeta$ is diffeomorphic to a local abelian fibration with central singular fiber the toroidal core $\mathcal{C}_A^\zeta \rightsquigarrow$ perverse filtration on $H^*(\mathcal{C}_A^\zeta)$

Conjecture (de Cataldo-Hausel-Migliorini, 2007)

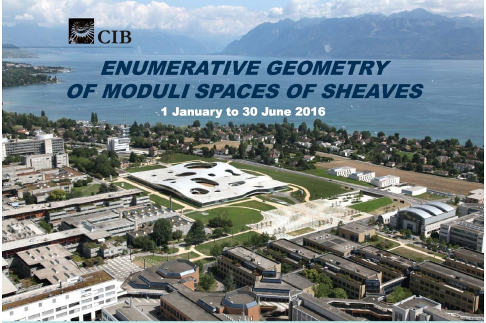
$\zeta \in \mathbb{T}_{\mathbb{R}}^d \subset \mathbb{T}_{\mathbb{C}}^d \rightsquigarrow W_{2k}H^*(\mathcal{M}_{\mathrm{B}}^\zeta) \cong P_k(H^*(\mathcal{C}_A^\zeta))$

- would follow from Mayer-Vietoris if we had $\mathcal{C}_A^\zeta := \chi_A^{-1}(\Delta_{bd}) \sim \mathcal{M}_{\mathrm{B}}^\zeta$ when $\zeta \in (\mathbb{R}^\times)^d \subset \mathbb{T}^d$ $\Delta_{bd} \subset \mathbb{R}^{n-d}$ bounded complex of the hyperplane arrangement

Problem

Can one cover the usual GL_n-character varieties $\mathcal{M}_{\mathrm{B}}$ with the (toric) character varieties corresponding to integral (nodal) spectral curves?

Program in Algebraic Geometry, CIB, EPF Lausanne



Semester organizers: T. Hausel (EPFL), R. Pandharipande (ETHZ), A. Szenes (Université de Genève) and F. Rodriguez Villegas (ICTP)

Higgs bundles and Hitchin system
Organizers: O. Garcia-Prada, T. Hausel, A. Szenes
Workshop: 11-15 January

Arithmetic aspects of moduli spaces
Organizers: T. Hausel, E. Letellier, F. Rodriguez Villegas
School: 25-29 January
Workshop: 1-5 February

Global singularity theory and curves
Organizers: R. Rimányi, A. Szenes
Workshop: May 9-13

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Organizers: T. Bridgeland, T. Hausel, B. Szendrői
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Organizers: D. Maulik, R. Pandharipande, V. Shende
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Organizers: J. Bryan, R. Pandharipande, R. Thomas
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