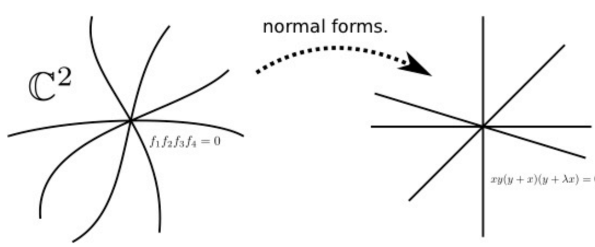
Yohann Genzmer: "The Zariski problem for homogeneous and quasi-homogeneous curves."

The *Zariski problem* concerns the analytical classification of germs of curves of the complex plane $\mathbb{C}^2$. In full generality, it is asked to understand as accurately as possible the quotient $\mathfrak{M}(f_0)$ of the topological class of the germ of curve $\{f_0(x,y)=0\}$ up to analytical equivalence relation.

The general approach of this problem consists in providing what is called some *normal forms*, which means, to provide a list of determined elements of the topological class of $\{f_0(x,y)=0\}$ such that each analytical class is represented by one and only one element of the list. Obviously, the list is in general infinite for a given topological class and parametrized by a set of continuous parameters. But this set, viewed as *the space of moduli* of the germ of curve $\{f_0(x,y)=0\}$ is of a great interest.

There are a lot of different possibilities in the choices of normal forms and nothing but your own taste guides you. For instance, the germ of curve given by $\{(x+\cdots)(y+\cdots)=0\}$ which consists in the union of two smooth and transversal germs of curves is analytically equivalent to $\{xy=0\}$, the union of germs of the axes. We will consider that the latter curve is a normal form for any union of two smooth and transversal germs of curves. Thus, we could also consider $\{(x+y)y=0\}$ but the expression $\{xy=0\}$ looks like much more simple and definitive, isn'it? The figure below illustrates exactly the same thing but in the case of four smooth and transversal germs of curves.



Four germs of smooth curves two by two transversal and its normal form $xy(y+x)(y+\lambda x)=0$ *parametrized by* λ.

When f_0 is irreducible, the approach of Zariski himself was to construct normal forms for the *Puiseux expansion* of the curve $\{f_0(x,y)=0\}$. Actually, if f_0 is irreducible, the curve can be locally parametrized by *its Puiseux parametrization*

$$\begin{cases} x &= t^m \\ y &= t^n+\sum_{i\geq n+1} a_i t^i \end{cases} \qquad t\in(\mathbb{C},0).$$

where $m<n$ and the a_i's are complex numbers. Then, the problem reduces to apply changes of coordinates in x, y and t in order to annihilate as much terms as possible in the series defining the coordinate y. For instance, in [5], among many other computations, Zariski shows that if $m=4$ and $n=5$ then, up two some changes of coordinates, there are only two possibilities

$$\begin{cases} x &= t^4 \\ y &= t^5 \end{cases} \text{ or } \begin{cases} x &= t^4 \\ y &= t^5+t^7 \end{cases}.$$

In other words, in the series defining y, one can annihilate all the terms except maybe the one of t^7 that can be normalized to 1 or 0. The above list consists in a set of normal forms for the topological class of the germ of curve $x^5 + y^4 = 0$.

In [5], the computations of Zariski are really clever but quite involved and generally only adapted to the studied case. Therefore, according to Zariski himself, the general case appeared at that time more or less inaccessible. Recently, this problem has known major progress with the work of A. Hefez and M. Hernandez [1, 2, 3]: they achieved the program of Zariski when f_0 is irreducible. This is a kind of *tour de force*, made possible probably by their computationnal and algorithmic point of view. However, as far as I understand, in the reducible case, the combinatorial size of their approach increases so fast with the number of branches that a different path is mandatory.

With E. Paul, we proposed a new approach also based upon the concept of normal forms but in which we work directly with the equation of the curve rather than with its Puiseux parametrization. Indeed, an expression such as $\{f_0(x, y) = 0\}$ is not only a curve but underlies also a function $-f_0-$ and a foliation - given by the $1-$form $\mathrm{d}f_0$ -. For the latter, we can adress the problem introduced at the very beginning of this abstract and ask whether or not is possible to describe the quotient $\mathfrak{M}(\mathrm{d}f_0)$ of the topological class of the germ of *foliation* $\mathrm{d}f_0$ up to analytical equivalence relation. There is a natural map

$$\pi : \mathfrak{M}(\mathrm{d}f_0) \longrightarrow \mathfrak{M}(f_0)$$

which happens to be onto. Our strategy is now in one sentence: it should be easier to understand the space $\mathfrak{M}(\mathrm{d}f_0)$ - which has *a priori* a simple structure - and the map above than the space $\mathfrak{M}(f_0)$ directly. When f_0 is quasi-homogeneous[1], we were able to prove the following two facts:

- $\mathfrak{M}(\mathrm{d}f_0)$ is a Zariski affine open set which can be explicitely described from topological datas associated to f_0.
- $\mathfrak{M}(\mathrm{d}f_0)$ is foliated by an explicit algebraic foliation whose *space of leaves* is identified with $\mathfrak{M}(f_0)$. Moreover, π is just the quotient map of this foliation.

For instance, when $f_0 = y^4 + x^5$ then $\mathfrak{M}(\mathrm{d}f_0)$ is the complex line $\mathbb{C}$ and the algebraic foliation mentioned above is generated by the vector field $z\frac{\partial}{\partial z}$. Thus the map π reduces in that case to the map

$$\mathbb{C} \to \{\overline{0}, \overline{\mathbb{C}^*}\}$$

and $\mathfrak{M}(f_0)$ consists in a set of two points with a non-separated topology: one point is closed, the other one is dense. Moreover, the normal forms for the germs of curves topologically equivalent to $y^4 + x^5 = 0$ are the two curves

$$y^4 + x^5 = 0 \qquad \text{or} \qquad y^4 + x^5 + y^2x^3 = 0.$$

This first one corresponding to the closed point is also the quasi-homogeneous one. This claim has to be compared with the one of Zariski mentioned above.

Finally, the study of the quotient $\mathfrak{M}(\mathrm{d}f_0)$ falls of a *foliated* tool called the *unfolding* of foliation, which allows us to understand the deformations of foliations preserving its topological structure and which was introduced by J.-F. Mattei in [4].

[1]Quasi-homogeneous means that there exists a vector field X such that $X \cdot f_0 = f_0$. In that case, there exist some adapted coordinates such that

$$f_0 = \sum_{ik+jl=d} a_{ij}x^iy^j \qquad k \wedge l = 1$$

The aim of the talk is to review, as far as possible, the approach of Zariski as well as the recent developments.

References

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