Invariants of Determinantal Varieties

Maria Aparecida Soares Ruas
ICMC-USP

Geometry of singular spaces and mappings
Luminy, March 4, 2015

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.
3. Invariants of Determinantal Singularities.

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.
3. Invariants of Determinantal Singularities.
4. Nash transformation of an EIDS

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.
3. Invariants of Determinantal Singularities.
4. Nash transformation of an EIDS
5. Sections of EIDS.

Introduction

We review basic results on determinantal varieties and show how to apply methods of singularity theory of matrices to study their invariants and geometry.

1. Essentially Isolated Determinantal Singularities (EIDS).
2. Singularity theory of matrices.
3. Invariants of Determinantal Singularities.
4. Nash transformation of an EIDS
5. Sections of EIDS.
6. Euler obstruction of EIDS.

Recent PhD. Thesis on Determinantal Varieties.

- Miriam Silva Pereira, ICMC, 2010.
  http://www.teses.usp.br/teses/disponiveis/55/55135/tde-22062010-133339/pt-br.php
- Brian Pike, North Carolina University, 2010.
  http://www.brianpike.info/thesis.pdf
- Bruna Oréfice Okamoto, UFSCar, 2011
  http://www.dm.ufscar.br/ppgm/attachments/article/179/download.pdf
- Nancy Carolina Chachapoyas Siesquén, ICMC and Université Aix Marseille, 2014.

Recent PhD. Thesis on Determinantal Varieties.

- Miriam Silva Pereira, ICMC, 2010. http://www.teses.usp.br/teses/disponiveis/55/55135/tde-22062010-133339/pt-br.php
- Brian Pike, North Carolina University, 2010. http://www.brianpike.info/thesis.pdf
- Bruna Oréfice Okamoto, UFSCar, 2011 http://www.dm.ufscar.br/ppgm/attachments/article/179/download.pdf
- Nancy Carolina Chachapoyas Siesquén, ICMC and Université Aix Marseille, 2014.

• W. Ebeling and S. M. Gusein-Zade, *On indices of* 1*-forms on determinantal singularities*, *Singularities and Applications*, **267**, 119-131, (2009).

Related recent new results on isolated determinantal singularities.

- J. J. Nuño-Ballesteros, B. Oréfice Okamoto, J. N. Tomazella, *The vanishing Euler characteristic of an isolated determinantal singullarity*, *Israel J. Math.*, **197** (2013), no. 1, 475-495.
- J. J. Nuño-Ballesteros, B. Oréfice Okamoto, J. N. Tomazella, *Equisingularity of families of isolated determinantal singularities*, *Math. Z.* to appear.
- T. Gaffney and A. Rangachev, *Pairs of modules and determinantal isolated singularities*, $arXiv : 1501.00201$.
- A. Frühbis-Krüger and M. Zach, *On the vanishing topology of isolated Cohen-Macaulay codimension 2 singularities*, preprint.

Generic determinantal variety

Definition

Let $M_{m,n}$ be the set of all $m \times n$ matrices with complex entries, and for all $t \leq \min\{m, n\}$ let

$$M^t_{m,n} = \{A \in M_{m,n} | rank(A) < t\}.$$

Generic determinantal variety

Definition

Let $M_{m,n}$ be the set of all $m \times n$ matrices with complex entries, and for all $t \leq \min\{m, n\}$ let

$$M^t_{m,n} = \{A \in M_{m,n} | rank(A) < t\}.$$

This set is a singular variety, called generic determinantal variety.

1. $M^t_{m,n}$ has codimension $(n - t + 1)(m - t + 1)$ in $M_{m,n}$
2. The singular set of $M^t_{m,n}$ is $M^{t-1}_{m,n}$
3. $M^t_{m,n} = \cup_{i=1,\ldots,t}(M^i_{m,n} \setminus M^{i-1}_{m,n})$, this partition is a Whitney stratification of $M^t_{m,n}$.

Determinantal varieties

Let $F : U \subset \mathbb{C}^N \to M_{m,n}$. For each x, $F(x) = (f_{ij}(x))$ is a $m \times n$ matrix; the coordinates f_{ij} are complex analytic functions on U.

Determinantal varieties

Let $F : U \subset \mathbb{C}^N \to M_{m,n}$. For each $x,\ F(x) = (f_{ij}(x))$ is a $m \times n$ matrix; the coordinates f_{ij} are complex analytic functions on U.

Definition

A determinantal variety of type (m, n, t), in an open domain $U \subset \mathbb{C}^N$ is a variety X that satisfies:

- *X is the preimage of the variety $M^t_{m,n}$. That is $X = F^{-1}(M^t_{m,n})$.*
- *$codim(X) = (m - t + 1)(n - t + 1)$ in $\mathbb{C}^N$*

Determinantal varieties

Example

Determinantal surface Let F be the following map:

$$\begin{array}{cccc} F & : & \mathbb{C}^4 & \rightarrow & M_{2,3} \\ & & (x,y,z,w) & \mapsto & \begin{pmatrix} z & y & x \\ w & x & y \end{pmatrix} \end{array}$$

Then $X = F^{-1}(M^2_{2,3}) = V(zx - wy, zy - wx, y^2 - x^2)$, X is a surface in $\mathbb{C}^4$ with isolated singularity at the origin.

Essentially Isolated Determinantal Singularities (EIDS)

The *Essential Isolated Determinantal Singularities (EIDS)* were defined by Ebeling and Gusein-Zade in [Proc. Steklov Inst. Math. (2009)].

Definition EIDS:

A germ $(X, 0) \subset (\mathbb{C}^N, 0)$ of a determinantal variety of type (m, n, t) has an essentially isolated determinantal singularity at the origin (EIDS) if F is transverse to all strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the stratification of $M^t_{m,n}$ in a punctured neighbourhood of the origin.

Essentially Isolated Determinantal Singularities (EIDS)

The *Essential Isolated Determinantal Singularities (EIDS)* were defined by Ebeling and Gusein-Zade in [Proc. Steklov Inst. Math. (2009)].

Definition EIDS:

A germ $(X,0) \subset (\mathbb{C}^N,0)$ of a determinantal variety of type (m,n,t) has an essentially isolated determinantal singularity at the origin (EIDS) if F is transverse to all strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the stratification of $M^t_{m,n}$ in a punctured neighbourhood of the origin.

The singular set of an EIDS $X = F^{-1}(M^t_{m,n})$ is the EIDS $F^{-1}(M^{t-1}_{m,n})$.

Example

An ICIS is an EIDS of type $(1, n, 1)$

More generally, $n \times (n+1)$ matrices with entries in $\mathcal{O}_N$ give a presentation of Cohen-Macaulay varieties of codimension 2 (Hilbert-Burch theorem).

Example

An ICIS is an EIDS of type $(1, n, 1)$

Example

The determinantal variety represented by the matrix

$$N = \begin{pmatrix} z & y & x \\ 0 & x & y \end{pmatrix}$$

is a curve in $\mathbb{C}^3$.

Example

An ICIS is an EIDS of type $(1, n, 1)$

Example

The determinantal variety represented by the matrix

$$N = \begin{pmatrix} z & y & x \\ 0 & x & y \end{pmatrix}$$

is a curve in $\mathbb{C}^3$.

More generally, $n \times (n+1)$ matrices with entries in $\mathcal{O}_N$ give a presentation of Cohen-Macaulay varieties of codimension 2 (Hilbert-Burch theorem).

Deformations of EIDS

Deformations (in particular, smoothings) of determinantal singularities which are themselves determinantal ones.

Deformations of EIDS

Deformations (in particular, smoothings) of determinantal singularities which are themselves determinantal ones.

Definition

(Ebeling and Gusein Zade (2009)) *An* *essential smoothing* $\tilde{X}$ *of the EIDS* $(X, 0)$ *is a subvariety lying in a neighbourhood* U *of the origin in* $\mathbb{C}^N$ *and defined by a perturbation* $\tilde{F} : U \to M_{m,n}$ *of the germ* F *such that* $\tilde{F}$ *is transversal to all the strata* $M^i_{m,n} \setminus M^{i-1}_{m,n}$*, with* $i \leq t$*.*

Deformations of EIDS

Deformations (in particular, smoothings) of determinantal singularities which are themselves determinantal ones.

Definition

(Ebeling and Gusein Zade (2009)) *An* *essential smoothing* $\tilde{X}$ *of the EIDS* $(X, 0)$ *is a subvariety lying in a neighbourhood* U *of the origin in* $\mathbb{C}^N$ *and defined by a perturbation* $\tilde{F} : U \to M_{m,n}$ *of the germ* F *such that* $\tilde{F}$ *is transversal to all the strata* $M^i_{m,n} \setminus M^{i-1}_{m,n}$, *with* $i \leq t$.

Example

For generic values of a, b, c, $\tilde{N}$ gives a smoothing of the curve in $\mathbb{C}^3$.

$$\tilde{N} = \begin{pmatrix} z & y+a & x+b \\ c & x & y \end{pmatrix}$$

Isolated determinantal singularities (IDS)

Proposition

- *An EIDS* $(X,0) \subset (\mathbb{C}^N,0)$ *of type* (m,n,t)*, defined by* $F:(\mathbb{C}^N,0) \to (M_{m,n},0)$ *has an isolated singularity at the origin if and only if* $N \leq (m-t+2)(n-t+2)$.
- $(X,0)$ *has a smoothing if and only if* $N < (m-t+2)(n-t+2)$.

Example

$F:\mathbb{C}^N \to M_{2,3},\;\; N \geq 6, F \pitchfork M^i_{2,3},\; i=1,2.$

$$F(x) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

When $N=6$, the singularity of $X=F^{-1}(M^2_{2,3})$ is isolated and X has no smoothing.

Matrices Singularity Theory

The methods of singularity theory apply both to real and complex matrices.

Matrices Singularity Theory

The methods of singularity theory apply both to real and complex matrices.

- Arnol'd (1971), square matrices.
- Bruce (2003), simple singularities of symmetric matrices.
- Bruce and Tari (2004), simple singularities of square matrices.
- Haslinger (2001), simple skew-symmetric.
- Frühbis-Krüger (2000) and Frühbis-Krüger and Neumer (2010), Cohen-Macaulay codimension 2 simple singularities.
- Goryunov and Mond (2005), Tjurina and Milnor numbers of square matrices.

Matrices Singularity Theory

The methods of singularity theory apply both to real and complex matrices.

- Arnol'd (1971), square matrices.
- Bruce (2003), simple singularities of symmetric matrices.
- Bruce and Tari (2004), simple singularities of square matrices.
- Haslinger (2001), simple skew-symmetric.
- Frühbis-Krüger (2000) and Frühbis-Krüger and Neumer (2010), Cohen-Macaulay codimension 2 simple singularities.
- Goryunov and Mond (2005), Tjurina and Milnor numbers of square matrices.
- M. Silva Pereira (2010), singularity theory of general $n \times m$ matrices.

A group $\mathcal{G}$ acting on the space of map-germs $F : (\mathbb{C}^N, 0) \to M_{M,n}$

$$\mathcal{R} = \{h : (\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0), \text{germs of analytic diffeomorphisms}\}$$

A group $\mathcal{G}$ acting on the space of map-germs $F : (\mathbb{C}^N, 0) \to M_{M,n}$

$$\mathcal{R} = \{h : (\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0), \text{germs of analytic diffeomorphisms}\}$$

$\mathcal{H} = GL_m(\mathcal{O}_N) \times GL_n(\mathcal{O}_N)$ and $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ (semi-direct product)

Definition

Given two matrices $F_1(x) = (f^1_{ij}(x))_{m\times n}$ and $F_2(x) = (f^2_{ij}(x))_{m\times n}$, we say that

$$F_1 \sim F_2 \text{ if } \exists\ (\phi, R, L) \in \mathcal{G} \text{ such that } F_1 = L^{-1}(\phi^* F_2)R.$$

A group $\mathcal{G}$ acting on the space of map-germs $F : (\mathbb{C}^N, 0) \to M_{M,n}$

$$\mathcal{R} = \{h : (\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0), \text{germs of analytic diffeomorphisms}\}$$

$\mathcal{H} = GL_m(\mathcal{O}_N) \times GL_n(\mathcal{O}_N)$ and $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ (semi-direct product)

Definition

Given two matrices $F_1(x) = (f^1_{ij}(x))_{m\times n}$ and $F_2(x) = (f^2_{ij}(x))_{m\times n}$, we say that

$$F_1 \sim F_2 \text{ if } \exists\ (\phi, R, L) \in \mathcal{G} \text{ such that } F_1 = L^{-1}(\phi^* F_2)R.$$

Proposition

If $F_1 \sim F_2$ then the corresponding determinantal varieties $X_1^t = F_1^{-1}(M^t_{m,n})$ and $X_2^t = F_2^{-1}(M^t_{m,n})$, $1 \le t \le m$ are isomorphic.

The group $\mathcal{G}$ is a geometric subgroup of the contact group $\mathcal{K}$. Hence the infinitesimal methods of singularity theory applies.

The group $\mathcal{G}$ is a geometric subgroup of the contact group $\mathcal{K}$. Hence the infinitesimal methods of singularity theory applies.

Definition

$F : U \to M_{m,n}$ is $\mathcal{G}$-stable if $T_e\mathcal{G}(F) = \Theta(F)$.

The above condition holds if and only if F is transversal to the canonical stratification of the space $M_{m,n}$.

Definition

The germ $F : (\mathbb{C}^N, 0) \to M_{m,n},\ F(x) = (f_{ij}(x))$ *is* $k - \mathcal{G}$*-finitely determined if for every* $G : (\mathbb{C}^N, 0) \to M_{m,n},\ G(x) = (g_{ij}(x))$ *such that* $j^k f_{ij}(x) = j^k g_{ij}(x),\ 1 \leq i \leq m,\ 1 \leq j \leq n$*, then* $G \sim F$.

Theorem

(M.S. Pereira, PhD thesis) *F is $\mathcal{G}$-finitely determined if and only if the Tjurina number of F*

$$\tau(F) = \ dim_{\mathbb{K}} \frac{\Theta(F)}{T_e\mathcal{G}(F)}$$

is finite

Definition

The germ $F:(\mathbb{C}^N,0)\to M_{m,n},\ \ F(x)=(f_{ij}(x))$ *is* $k-\mathcal{G}$*-finitely determined if for every* $G:(\mathbb{C}^N,0)\to M_{m,n},\ \ G(x)=(g_{ij}(x))$ *such that* $j^kf_{ij}(x)=j^kg_{ij}(x),\ 1\leq i\leq m,\ 1\leq j\leq n,$ *then* $G\sim F$.

Theorem

(M.S. Pereira, PhD thesis) *F is $\mathcal{G}$-finitely determined if and only if the Tjurina number of F*

$$\tau(F)=\ dim_{\mathbb{K}}\frac{\Theta(F)}{T_e\mathcal{G}(F)}$$

is finite

In this case, F has a versal unfolding with τ-parameters.

Theorem

(M.S. Pereira, PhD thesis) *(Geometric criterion of finite determinacy)* F *is finitely* $\mathcal{G}$*-determined if and only if there exists a representative* $F : U \to M_{m,n}$ *such that for all* $x \neq 0$ *in* U, $rankF(x) + 1 = i$, *then* F *is transversal to* $M^i_{m,n}$ *at* x.

Theorem

(M.S. Pereira, PhD thesis) *(Geometric criterion of finite determinacy) F is finitely $\mathcal{G}$-determined if and only if there exists a representative $F : U \to M_{m,n}$ such that for all $x \neq 0$ in U, $rankF(x) + 1 = i$, then F is transversal to $M^i_{m,n}$ at x.*

F is $\mathcal{G}$-finitely determined if and only if $X = F^{-1}(M^t_{m,n})$ is an EIDS for all $t \leq \min\{m, n\}$.

A stable perturbation $\tilde{F}$ of F defines an essential smoothing $\tilde{X} = \tilde{F}^{-1}(M^m_{m,n})$ of X.

Example

Let

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

Example

Let

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

This is the first normal form of the classification of simple Cohen-Macaulay singularities of codimension 2 of A. Fübhis-Kruger and A. Neumer in [Comm. Alg. 38, 454-495, (2010)].

The surface $X_k \subset \mathbb{C}^4$ associated to A_k is defined by the ideal $\left\langle xz^k - yw, x^2 - zw, xy - z^{k+1} \right\rangle$.

Example

Let

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

This is the first normal form of the classification of simple Cohen-Macaulay singularities of codimension 2 of A. Fübhis-Kruger and A. Neumer in [Comm. Alg. 38, 454-495, (2010)].

The surface $X_k \subset \mathbb{C}^4$ associated to A_k is defined by the ideal $\langle xz^k - yw, x^2 - zw, xy - z^{k+1} \rangle$.

The versal unfolding of F_k is

$$\tilde{F}_k(x, y, z, w, u_0, u_1, \ldots, u_k) = \begin{pmatrix} x & y & z \\ w & z^k + \Sigma_0^{k-1} u_i z^i & x + u_k \end{pmatrix},$$

$$\tau(F_k) = k + 1.$$

Example

Let

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

This is the first normal form of the classification of simple Cohen-Macaulay singularities of codimension 2 of A. Fübhis-Kruger and A. Neumer in [Comm. Alg. 38, 454-495, (2010)].

The surface $X_k \subset \mathbb{C}^4$ associated to A_k is defined by the ideal $\left\langle xz^k - yw, x^2 - zw, xy - z^{k+1} \right\rangle$.

The versal unfolding of F_k is

$$\tilde{F}_k(x, y, z, w, u_0, u_1, \ldots, u_k) = \begin{pmatrix} x & y & z \\ w & z^k + \Sigma_0^{k-1} u_i z^i & x + u_k \end{pmatrix},$$

$$\tau(F_k) = k + 1.$$

Singular fibration of an EIDS

$$F : (\mathbb{C}^N, 0) \to M_{m,n}, \ (X, 0) = F^{-1}(M^t_{m,n})$$

$$\widetilde{F} : W \subset \mathbb{C}^N \times \mathbb{C}^s \to M_{m,n}, \widetilde{F}(x, 0) = F(x), \ \widetilde{F} \pitchfork \{M^i_{m,n} \backslash M^{i-1}_{m,n}\}, \ \mathfrak{X} = \widetilde{F}^{-1}(M$$

$$\begin{array}{ccc} \mathfrak{X} & \subset & W \subset \mathbb{C}^N \times \mathbb{C}^s \\ & & \downarrow \pi \\ B(F) & \subset & \mathbb{C}^s \end{array},$$

where $B(F)$ is the bifurcation set.

Singular fibration of an EIDS

$$F : (\mathbb{C}^N, 0) \to M_{m,n},\ (X, 0) = F^{-1}(M^t_{m,n})$$

$$\widetilde{F} : W \subset \mathbb{C}^N \times \mathbb{C}^s \to M_{m,n}, \widetilde{F}(x, 0) = F(x),\ \widetilde{F} \pitchfork \{M^i_{m,n} \backslash M^{i-1}_{m,n}\},\ \mathfrak{X} = \widetilde{F}^{-1}(M$$

$$\begin{array}{ccc} \mathfrak{X} & \subset & W \subset \mathbb{C}^N \times \mathbb{C}^s \\ & & \downarrow \pi \\ B(F) & \subset & \mathbb{C}^s \end{array},$$

where $B(F)$ is the bifurcation set.

For $u \in \mathbb{C}^s \setminus B(F)$, $\widetilde{F}_u$ defines $\widetilde{X}_u$ which is an essential smoothing of X.

Singular fibration of an EIDS

$$F : (\mathbb{C}^N, 0) \to M_{m,n}, \; (X, 0) = F^{-1}(M^t_{m,n})$$

$$\widetilde{F} : W \subset \mathbb{C}^N \times \mathbb{C}^s \to M_{m,n}, \widetilde{F}(x, 0) = F(x), \; \widetilde{F} \pitchfork \{M^i_{m,n} \backslash M^{i-1}_{m,n}\}, \; \mathfrak{X} = \widetilde{F}^{-1}(M$$

$$\begin{array}{ccc} \mathfrak{X} & \subset & W \subset \mathbb{C}^N \times \mathbb{C}^s \\ & & \downarrow \pi \\ B(F) & \subset & \mathbb{C}^s \end{array},$$

where $B(F)$ is the bifurcation set.

For $u \in \mathbb{C}^s \setminus B(F)$, $\widetilde{F}_u$ defines $\widetilde{X}_u$ which is an essential smoothing of X. The generic fibre $\widetilde{X}_u$ is well defined.

Invariants of EIDS

Definition

(Damon and Pike [Geom. Topol., **18**(2) (2014)], Ebeling and Gusein-Zade (2009)) *The* *singular vanishing Euler characteristic of X*, *is defined as*

$$\tilde{\chi}(X) = \tilde{\chi}(\widetilde{X}_u) = \chi(\widetilde{X}_u) - 1.$$

(Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J. Math. **197** (2013), 475-495.]) *When $\widetilde{X}_u$ is smooth, vanishing Euler characteristic of X is*

$$\nu(X) = (-1)^{dim(X)}(\chi(\widetilde{X}_u) - 1).$$

Let X and $\mathfrak{X}$ be as above, $\dim(X) = d$.

Definition: The d-polar multiplicity, (Gaffney [Top. (1993)])

Let $p : X \to \mathbb{C}$, with isolated singularity. Let

$$\pi : \mathfrak{X} \subset \mathbb{C}^N \times \mathbb{C} \to \mathbb{C},$$

$\pi^{-1}(0) = X$, $\widetilde{p} : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}$ linear projection, $\widetilde{p}(x, 0) = p(x)$, and for all $t \neq 0, \widetilde{p}_t(.)$ is a generic deformation of p.
Let

$$P_d(X, \pi, p) = \overline{\Sigma(\pi, \widetilde{p})|\mathfrak{X}_{reg}}$$

be the relative polar variety of X relative to π and p.
Define

$$m_d(X, \pi, p) = m_0(P_d(\pi, p)).$$

In general, $m_d(X, \pi, p)$ depends on the choices of $\mathfrak{X}$ and p, but when X is an EIDS, m_d depends only on X and p. Furthermore, if p is a generic linear embedding, m_d is an invariant of the EIDS X, denoted by $m_d(X)$.

Proposition:

Let $X = F^{-1}(M_{m,n})$ and $\widetilde{X}$ its essential smoothing. Let $p : X \to \mathbb{C}$ be a function with isolated singularity in X. Then

$$m_d(X, p) = \#\,\text{non-degenerated critical points of}\, \widetilde{p}_t|(\widetilde{X}_t)_{\text{reg}},$$

where $\widetilde{p}_t$ is a generic perturbation of p (Morsification), and $\widetilde{X}_t$ an essential smoothing of X. When p is a generic linear function defined on X, we write $m_d(X, p) = m_d(X)$.

Nash transformation

Let X be a $d-$ dimensional analytic complex variety in $\mathbb{C}^N$.

$Gr(d,N)$ the Grassmannian of d-subspaces in $\mathbb{C}^N$.

Let $\pi : \mathbb{C}^N \times Gr(d,N) \to \mathbb{C}^N$ be the projection to the $\mathbb{C}^N$.

On the regular part of X, we have the Gauss map defined by:

Nash transformation

Let X be a $d-$ dimensional analytic complex variety in $\mathbb{C}^N$.

$Gr(d, N)$ the Grassmannian of d-subspaces in $\mathbb{C}^N$.

Let $\pi : \mathbb{C}^N \times Gr(d, N) \to \mathbb{C}^N$ be the projection to the $\mathbb{C}^N$.

On the regular part of X, we have the Gauss map defined by:

$$\begin{array}{rcccc} s & : & X_{reg} & \to & \mathbb{C}^N \times Gr(d, N) \\ & & x & \mapsto & (x, T_x X_{reg}) \end{array}$$

Definition

The Nash transformation $\widehat{X}$ of X *is the closure in* $\mathbb{C}^N \times Gr(d, N)$ *of the image of* s, *i.e.,*

$$\widehat{X} = \overline{\{(x, W) | x \in X_{reg}, W = T_x X_{reg}\}}.$$

Definition

The Nash transformation $\widehat{X}$ *of* X *is the closure in* $\mathbb{C}^N \times Gr(d, N)$ *of the image of* s*, i.e.,*

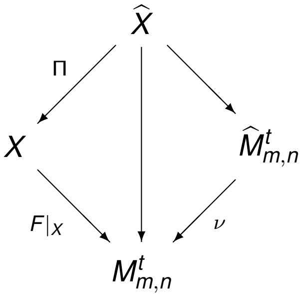
$$\widehat{X} = \overline{\{(x, W) | x \in X_{reg}, W = T_x X_{reg}\}}.$$

If $x \in X$ is a singular point, then the fibre over x:
$\nu^{-1}(x) = \{(x, T)/T = \lim_{x_n \to x} (T_{x_n} X), x_n \in X_{reg}\}, \ \nu = \pi|_{\widehat{X}}$

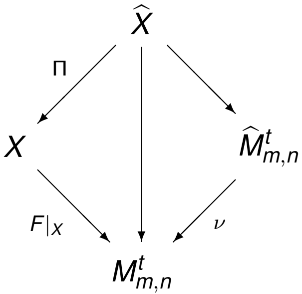
Proposition

(Arbarello, Cornalba, Griffiths and Harris) *The Nash transformation* $\widehat{M^t_{m,n}}$ *of* $M^t_{m,n}$, $1 \leq t \leq m$ *is smooth.*

Let X be an EIDS of type (m,n,t) in $U \subset \mathbb{C}^N$. The Nash transformation $\widehat{X}$ is the fibre product $\widehat{M}^t_{m,n} \times_{M^t_{m,n}} X$ of $\widehat{M}^t_{m,n}$ and X over $M^t_{m,n}$.



Let X be an EIDS of type (m, n, t) in $U \subset \mathbb{C}^N$. The Nash transformation $\widehat{X}$ is the fibre product $\widehat{M}^t_{m,n} \times_{M^t_{m,n}} X$ of $\widehat{M}^t_{m,n}$ and X over $M^t_{m,n}$.



If F is transversal to the canonical stratification in $M_{m,n}$, then $\widehat{X}$ is a resolution of X.

Theorem

(Chachapoyas-Siesquen, PhD thesis) *Let* $X = F^{-1}(M^t_{m,n}) \subset \mathbb{C}^N$ *be an EIDS, defined by* $F : U \subset \mathbb{C}^N \to M_{m,n}$.

If F *is transversal to all the limits of the tangent spaces to the strata of* $M^t_{m,n}$ *then* $\widehat{X}$ *is smooth.*

Theorem

(Chachapoyas-Siesquen, PhD thesis) *Let* $X = F^{-1}(M^t_{m,n}) \subset \mathbb{C}^N$ *be an EIDS, defined by* $F : U \subset \mathbb{C}^N \to M_{m,n}$.

If F *is transversal to all the limits of the tangent spaces to the strata of* $M^t_{m,n}$ *then* $\widehat{X}$ *is smooth.*

Questions

Does a finite iteration of Nash transformations resolve the singularities of an EIDS X?

Describe the singularities of $\widehat{X}$.

Anne Frühbis-Krüger and Mathias Zach in [*On the vanishing topology of isolated Cohen-Macaulay Codimension 2 singularities*],extended to Cohen-Macauly codimension singularities a technique called Tjurina modification, previously used by Tjurina for surfaces.

Anne Frühbis-Krüger and Mathias Zach in [*On the vanishing topology of isolated Cohen-Macaulay Codimension 2 singularities*],extended to Cohen-Macauly codimension singularities a technique called Tjurina modification, previously used by Tjurina for surfaces.

Using this tool, they relate a given Cohen-Macaulay Codimension 2 singularity to a local complete intersection scheme.

Anne Frühbis-Krüger and Mathias Zach in [*On the vanishing topology of isolated Cohen-Macaulay Codimension 2 singularities*],extended to Cohen-Macauly codimension singularities a technique called Tjurina modification, previously used by Tjurina for surfaces.

Using this tool, they relate a given Cohen-Macaulay Codimension 2 singularity to a local complete intersection scheme.

They characterize a class of $X \subset \mathbb{C}^5$ threefold Cohen-Macaulay Codimension 2 singularities whose Tjurina transformation Y only has isolated complete intersection singularities.

Isolated Determinantal Singularities admitting smoothing: N < (m-t+2)(n-t+2)

If $X \subset \mathbb{C}^N$ is a normal variety admitting smoothing, then $b_1(X_u) = 0$ (Greuel and Steenbrink [*Proc. Symp. Pure Math.* **40**,(1983).])
Determinantal isolated singularities are normal singularities, so this holds for them.

Theorem

(Nuno-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J. 2013]) *Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function and $\widetilde{X}$ an essential smoothing of X. Then*

$$\#\Sigma(p|_{\widetilde{X}}) = \nu(X, 0) + \nu(X \cap p^{-1}(0), 0),$$

where $\#\Sigma(p|_{\widetilde{X}})$ denotes the number of critical points of $p|_{\widetilde{X}}$.

Determinantal surfaces

M. S. Pereira and M. Ruas [Math. Scand., 2014], Nuno-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J., 2013], Damon and Pike [Geom. Topol. 2014].

Milnor number of determinantal surface in $\mathbb{C}^N$,

The Milnor number of X at 0, denoted by $\mu(X)$, is defined as $\mu(X) = b_2(X_u)$, where X_u is the generic fiber of X and $b_2(X_u)$ is the 2 -th Betti number.

Le-Greuel type formula

Proposition: [Math. Scand. 2014], [Israel J. 2013], [Geom. Top. 2014]

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be 2-dimensional IDS admitting smoothing. Let $p : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be a generic linear function on X. Then,

$$\mu(X) + \mu(X \cap p^{-1}(0)) = m_2(X),$$

where $m_2(X)$ is the second polar multiplicity of X.

Le-Greuel type formula

Proposition: [Math. Scand. 2014], [Israel J. 2013], [Geom. Top. 2014]

Let $(X,0) \subset (\mathbb{C}^N,0)$ be 2-dimensional IDS admitting smoothing. Let $p : (\mathbb{C}^N,0) \to (\mathbb{C},0)$ be a generic linear function on X. Then,

$$\mu(X) + \mu(X \cap p^{-1}(0)) = m_2(X),$$

where $m_2(X)$ is the second polar multiplicity of X.

M.S. Pereira'conjecture, Ph. Thesis (2010)

([Math. Scand. 2014], [Geom. Top. 2014])
If $X^2 \subset \mathbb{C}^4$ is a simple 2-dimensional IDS, then $\mu(X) + 1 = \tau(X)$

Le-Greuel type formula

Proposition: [Math. Scand. 2014], [Israel J. 2013], [Geom. Top. 2014]

Let $(X,0) \subset (\mathbb{C}^N,0)$ be 2-dimensional IDS admitting smoothing. Let $p : (\mathbb{C}^N,0) \to (\mathbb{C},0)$ be a generic linear function on X. Then,

$$\mu(X) + \mu(X \cap p^{-1}(0)) = m_2(X),$$

where $m_2(X)$ is the second polar multiplicity of X.

M.S. Pereira'conjecture, Ph. Thesis (2010)

([Math. Scand. 2014], [Geom. Top. 2014])
If $X^2 \subset \mathbb{C}^4$ is a simple 2-dimensional IDS, then $\mu(X) + 1 = \tau(X)$

Question

Does this formula hold for all 2-dimensional IDS ?

Cohen-Macaulay cod 2 threefold singularities.

- Damon and Pike, Geom. Top. **18**, 2014: describe a method to compute the difference of Betti numbers $b_2(\widetilde{X}) - b_3(\widetilde{X})$ for isolated Cohen- Macaulay 3-fold singularities X in $\mathbb{C}^5$.
- Frühbis-Krüger and Zach, 2015: use the Tjurina modification to compute the Betti numbers b_2 and b_3 of the essential smoothing of a 3-fold singularity.

Theorem (Frühbis-Krüger and Zach)

If the singularities of the Tjurina modification Y of a 3-fold Cohen-Macaulay cod. 2 singularity X are isolated complete intersections, then

$$b_0(X) = 1,\ b_1(X) = 0,\ b_2(X) = 1,\ b_3(X) = r,$$

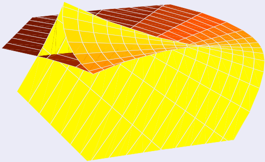
where r is the sum of Milnor numbers of the singularities of Y.

Sections of Determinantal Varieties

Definition

The hyperplane $H \subset \mathbb{C}^N$, given by the kernel of the linear function $p : \mathbb{C}^N \to \mathbb{C}$ is called general with respect to X at 0 if H is not the limit of tangent hyperplanes to X at 0.

Example: Swallowtail

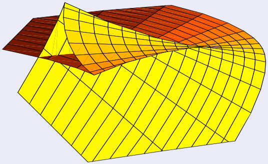


Sections of Determinantal Varieties

Definition

The hyperplane $H \subset \mathbb{C}^N$, given by the kernel of the linear function $p : \mathbb{C}^N \to \mathbb{C}$ is called general with respect to X at 0 if H is not the limit of tangent hyperplanes to X at 0.

Example: Swallowtail



Definition

Let $\{V_i\}$ be a stratification of $X^d \subset \mathbb{C}^N$. The hyperplane $H \subset \mathbb{C}^N$ is called strongly general at the origin if H is general and there exists a neighbourhood U of 0 such that for all strata V_i of X, with $0 \in \overline{V}_i$, then $H \pitchfork V_i$ at x, $\forall\, x \in U \backslash \{0\}$.

Proposition

[Chachapoyas-Siesquen] Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a d- dimensional EIDS of type (m, n, t). If $H \subset \mathbb{C}^N$ is a strongly general hyperplane then $X \cap H \subset \mathbb{C}^{N-1}$ is a $d-1$-dimensional EIDS of the same type.

Minimality of Milnor number

Let X be a d- dimensional complex variety. A hyperplane H is general if and only if $\mu(X \cap H)$ is minimum.

Minimality of Milnor number

Let X be a d- dimensional complex variety. A hyperplane H is general if and only if $\mu(X \cap H)$ is minimum.

1. B. Teissier [Astérisque, 7 et 8 , (1973)], Henry and Le Dung Trang, [LNM, 482 (1975)], the case of hypersurfaces.

Minimality of Milnor number

Let X be a d- dimensional complex variety. A hyperplane H is general if and only if $\mu(X \cap H)$ is minimum.

1. B. Teissier [Astérisque, 7 et 8 , (1973)], Henry and Le Dung Trang, [LNM, 482 (1975)], the case of hypersurfaces.
2. T. Gaffney [Travaux en Cours, **55** (1997)], ICIS.

Minimality of Milnor number

Let X be a d- dimensional complex variety. A hyperplane H is general if and only if $\mu(X \cap H)$ is minimum.

1. B. Teissier [Astérisque, 7 et 8 , (1973)], Henry and Le Dung Trang, [LNM, 482 (1975)], the case of hypersurfaces.
2. T. Gaffney [Travaux en Cours, **55** (1997)], ICIS.
3. J. Snoussi [Comment. Math. Helv. **76** (1), (2001)], normal surfaces in $\mathbb{C}^N$.

Minimality of Milnor number

Similar result holds for 3-dimensional EIDS.

Theorem

(Chachapoyas-Siésquen)*Let $X \subset \mathbb{C}^N$ be a 3- dimensional determinantal variety with isolated singularity and let H be a hyperplane in $\mathbb{C}^N$. Suppose that $X \cap H$ has an isolated singular point, then the following conditions are equivalent.*

- *H is general to X at 0.*
- *$\mu(X \cap H)$ is minimum and $\mu(X \cap H \cap H')$ is minimum for all hyperplane H' general to X and to $X \cap H$.*

Sections of EIDS

The following result is a generalization of a result of Lê Dung Trang [Singularity theory, World Sci. Publ., Hackensack, NJ, 2007.] We use the Lê-Greuel formula for surfaces.

Theorem

Let $X \subset \mathbb{C}^N$ be a d- dimensional EIDS. Let H, H' be hyperplanes in $\mathbb{C}^N$ strongly general to $(X, 0)$ at the origin. Then there exist P and P', $P \subset H$ and $P' \subset H'$ such that $\operatorname{codim} P = \operatorname{codim} P' = d - 2$, and the determinantal surfaces $X \cap P$ and $X \cap P'$ satisfy the following conditions:

- **a)** *$X \cap P$ and $X \cap P'$ have isolated singularity.*
- **b)** *$X \cap P$ and $X \cap P'$ admit smoothing.*
- **c)** $\mu(X \cap P) = \mu(X \cap P')$.

Euler obstruction

Theorem: (Brasselet, D. T. Lê, and J. Seade,[Topology 39, (2000)])

Let $(X,0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^N$. Let $\{V_i\}$ be a Whitney stratification of a small representative X of $(X,0)$. Then for a generic complex linear form $l:\mathbb{C}^N\to\mathbb{C}$, and for ϵ and $r\neq 0$ sufficiently small, the following formula for the Euler obstruction of $(X,0)$ holds,

$$Eu_0(X)=\sum_i \chi(V_i\cap B_\epsilon\cap l^{-1}(r))Eu_{V_i}(X),$$

where the sum is over strata V_i such that $0\in\overline{V}_i$ and $Eu_{V_i}(X)$ is the Euler obstruction of X in any point of V_i.

Euler obstruction of an EIDS: $N \leq (m-t+3)(n-t+3)$

(N. Chachapoyas-Siesquen's PhD. Thesis (2014))

Let $X = F^{-1}(M^t_{m,n})$ be an EIDS, defined by $F : \mathbb{C}^N \to M_{m,n}$. If $N \leq (m-t+3)(n-t+3)$ then the singular part $\Sigma X = F^{-1}(M^{t-1}_{m,n})$ is an IDS . The variety X admits 3 strata $\{V_0, V_1, V_2\}$, $V_0 = \{0\}$, $V_1 = \Sigma X \backslash \{0\}$, $V_2 = X_{reg}$. Then, we have

Euler obstruction of an EIDS: $N \leq (m-t+3)(n-t+3)$

(N. Chachapoyas-Siesquen's PhD. Thesis (2014))

Let $X = F^{-1}(M_{m,n}^{t})$ be an EIDS, defined by $F : \mathbb{C}^N \to M_{m,n}$. If $N \leq (m-t+3)(n-t+3)$ then the singular part $\Sigma X = F^{-1}(M_{m,n}^{t-1})$ is an IDS . The variety X admits 3 strata $\{V_0, V_1, V_2\}$, $V_0 = \{0\}$, $V_1 = \Sigma X \backslash \{0\}$, $V_2 = X_{reg}$. Then, we have

$$Eu_0(X) = \chi(\Sigma X \cap l^{-1}(r) \cap B_\epsilon)(\chi(L_{V_1}) - 1) + \chi(X \cap l^{-1}(r) \cap B_\epsilon).$$

where $l : \mathbb{C}^N \to \mathbb{C}$ is a generic linear projection centered at 0, L_{V_1} is the complex link of the stratum V_1 in X and B_ϵ is the ball of radius ϵ in $\mathbb{C}^N$.

Euler obstruction, $F : \mathbb{C}^N \to M_{2,3}$

Proposition: (N. Chachapoyas-Siesquen's PhD. Thesis)

Let $X = F^{-1}(M^2_{2,3}) \subset \mathbb{C}^N$ be an EIDS defined by the function $F : \mathbb{C}^N \to M_{2,3}$.

- If $N = 6$, then

$$Eu_0(X) = \chi(X \cap l^{-1}(r)) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)) + 1.$$

Euler obstruction, $F : \mathbb{C}^N \to M_{2,3}$

Proposition: (N. Chachapoyas-Siesquen's PhD. Thesis)

Let $X = F^{-1}(M^2_{2,3}) \subset \mathbb{C}^N$ be an EIDS defined by the function $F : \mathbb{C}^N \to M_{2,3}$.

- If $N = 6$, then

$$Eu_0(X) = \chi(X \cap l^{-1}(r)) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)) + 1.$$

- If $N \geq 7$, then

$$Eu_0(X) = (-1)^{N-7}\mu(\Sigma X \cap l^{-1}(0)) + \tilde{\chi}(X \cap l^{-1}(0)) + 2.$$

Euler obstruction, $F : \mathbb{C}^N \to M_{2,3}$

Proposition: (N. Chachapoyas-Siesquen's PhD. Thesis)

Let $X = F^{-1}(M^2_{2,3}) \subset \mathbb{C}^N$ be an EIDS defined by the function $F : \mathbb{C}^N \to M_{2,3}$.

- If $N = 6$, then

$$Eu_0(X) = \chi(X \cap l^{-1}(r)) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)) + 1.$$

- If $N \geq 7$, then

$$Eu_0(X) = (-1)^{N-7}\mu(\Sigma X \cap l^{-1}(0)) + \tilde{\chi}(X \cap l^{-1}(0)) + 2.$$

Proposition: (N. Chachapoyas-Siesquen's PhD. Thesis)

If F has corank 1, $N \geq 7$, then

$$Eu_0(X) = 2.$$

Thanks !