Unexpected fillings -- CIRM

Wednesday, September 8, 2021 11:32 AM

Unexpected fillings of links of rational surface singularities

Joint with Olga Plamenevskaya

Set up: Isolated complex surface singularity

Link of singularity:

3-manifold, contact structure

(Y, ξ)

Consider:

Milnor fibers (smoothings) $\subsetneq$ Minimal Symplectic fillings of (Y, ξ) (the link)

Question: For which singularities are there symplectic fillings of the link which are NOT Milnor fibers or the resolution?

"unexpected fillings"

Prior results:

[Ohta-Ono] : Simple and simple elliptic singularities

[Lisca, Nemethi-Popescu-Pampu] : cyclic singularities

[Bhupal-Ono, Park-Park-Shin-Urzua]: quotient singularities

No unexpected fillings

[Akhmedov-Ozbagci] : Higher genus non rational examples

$\exists$ infinitely many symplectic fillings with $b_1 \neq 0$

Lots of unexpected fillings

What class of singularities should we expect has no unexpected fillings?

A nice class of singularities to consider: Rational with reduced fundamental cycle (RFC)

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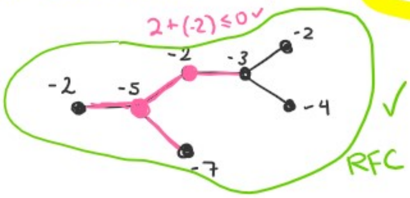
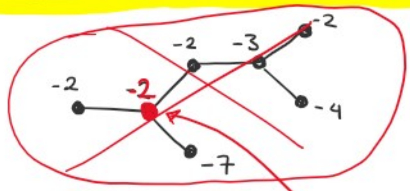
Artin's fundamental cycle $Z_{min} = \sum a_v E_v$ reduced means all $a_v = 1$.

In terms of minimal good resolution dual graph:

- the graph is a tree, each E_v genus 0
- For each vertex v, #edges adj. to v + normal Euler # ≤ 0





$3+(-5) = -2 \leq 0$ ✓

$3+(-2) = 1 \not\leq 0$ bad vertices

Cyclic singularities ⊂ RFC singularities ⊂ Rational singularities

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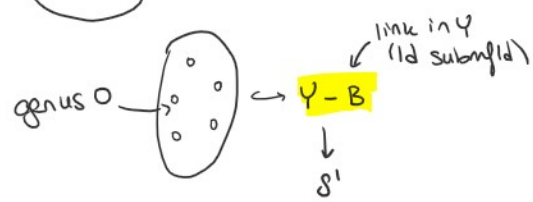
RFC singularities are also nice from contact/symplectic perspective:

Theorem [Ghiggini-Golla-Plamenevskaya]
A link of a normal surface singularity has a planar open book decomposition if and only if the singularity is RFC.



Theorem [Wendl] If (Y,ξ) has a planar open book decomposition with monodromy φ

minimal symplectic fillings of (Y,ξ) $\xleftrightarrow{1:1}$ factorizations of φ into positive Dehn twists

Every min sympl. filling of planar ct mfld has a planar Lefschetz fibration w/ bdry that open book.

Symplectic fillings are more limited... are unexpected fillings possible here?

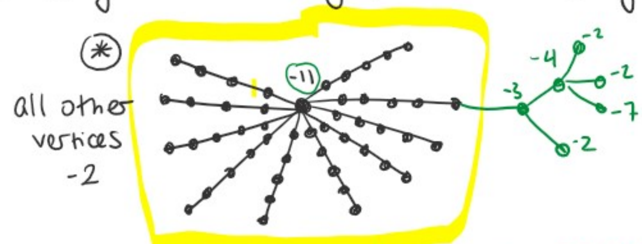
Theorem [Plamenevskaya-S.] There are RFC singularities with unexpected fillings!

For any $N > 0$, there exists an RFC singularity with $\geq N$ symplectic fillings which are not diffeomorphic rel boundary data to any Milnor fiber.

simply connected $b_1 = 0$

How prevalent is this among RFC singularities?

If the min. good resolution graph has a subgraph (*) there exists an unexpected filling



But many RFC singularities have no unexpected fillings:

Theorem [P-S] If all normal Euler numbers in min good res are ≤ -5, there is a unique symplectic filling which is the resolution/Artin smoothing.

How to prove it?

① Understand Milnor fibers better:

② Prove a symplectic analogue:

de Jong - van Straten:

Milnor fibers of an RFC singularity

$\updownarrow$ 1:1

Certain (sandwiched) complex deformations of a (decorated) complex plane curve singularity associated to the RFC singularity.

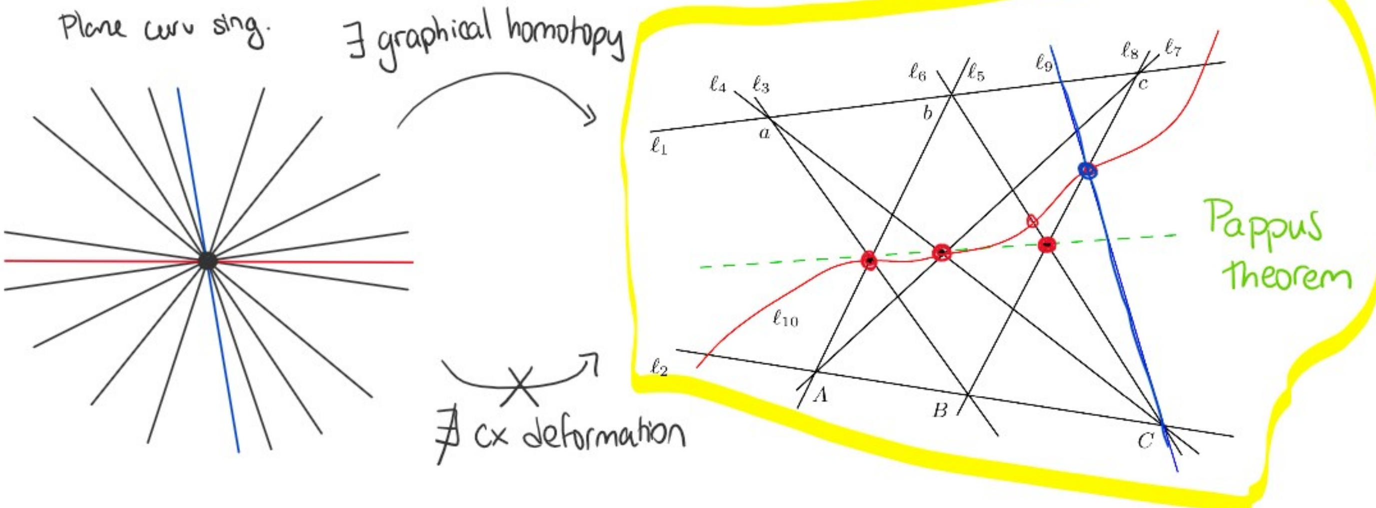
Plamenevskaya-S.:

Symplectic fillings of a link of an RFC sing

$\updownarrow$ 1:1

Curves obtained by graphical homotopies from the associated decorated plane curve sing.

Example:



⇝ gives rise to an RFC singularity with an unexpected filling

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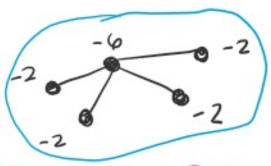
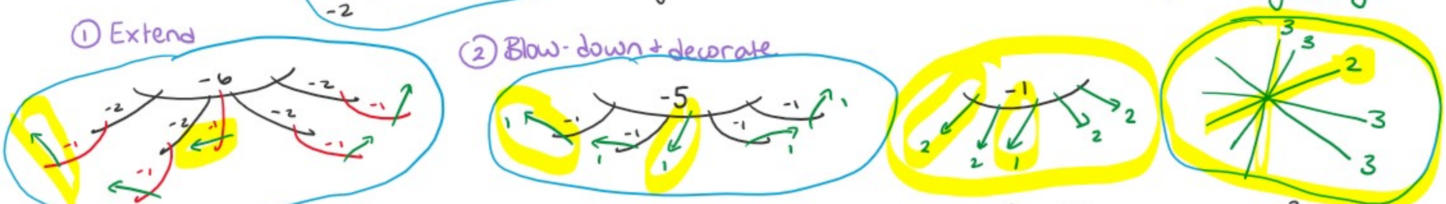
How to associate a decorated plane curve singularity to an RFC singularity: (dJvS)

Example 1:



← RFC singularity min good resolution → Decorated plane curve singularity:

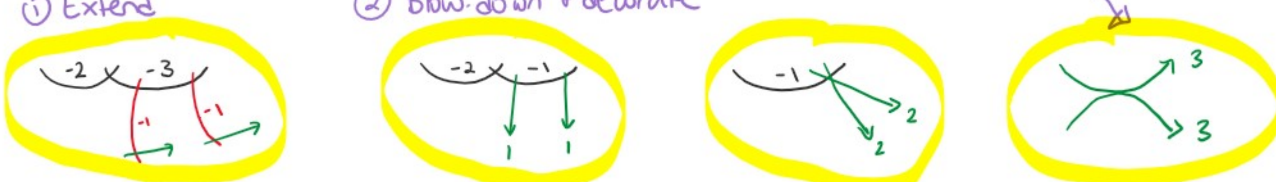
① Extend ② Blow-down + decorate



$\mathbb{C}^2 \# 10\overline{\mathbb{CP}^2}$ $\quad$ $\mathbb{C}^2 \# 5\overline{\mathbb{CP}^2}$ $\quad$ $\mathbb{C}^2 \# \overline{\mathbb{CP}^2}$ $\quad$ $\mathbb{C}^2$

Example 2:

① Extend ② Blow-down + decorate



Deformation rules: • Make all singularities transverse multi-points (marked points)

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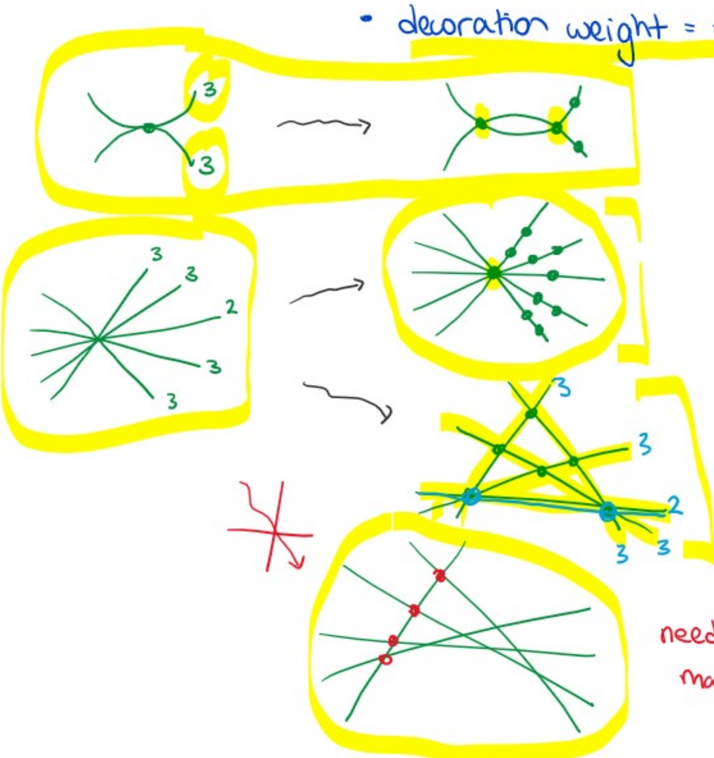
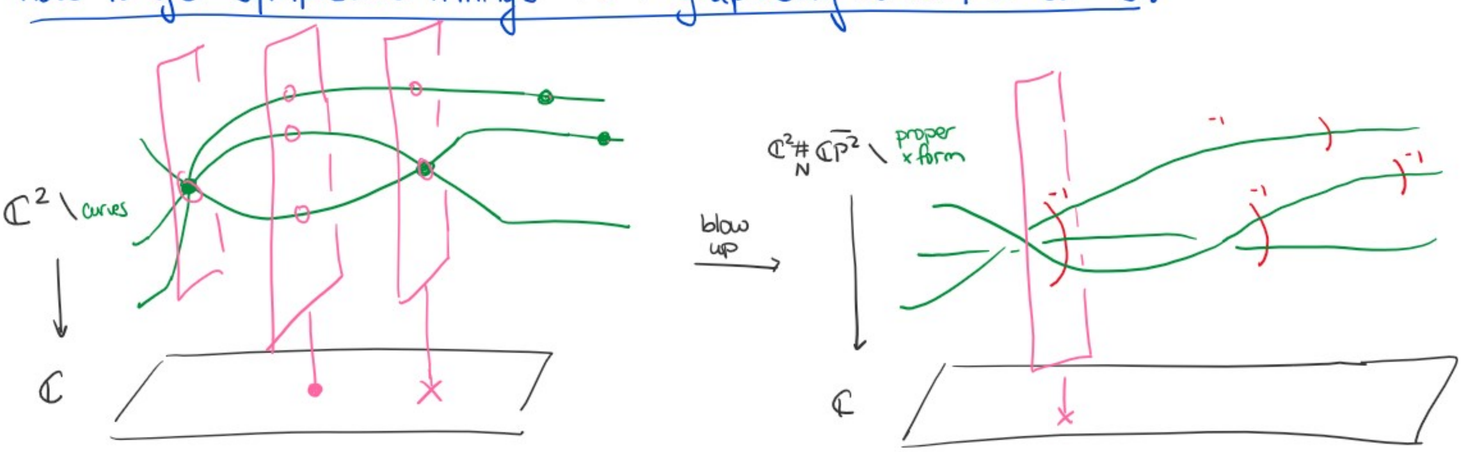
- Make all singularities transverse multi-points (marked points)
- decoration weight = # marked points (limits # of multipoints)

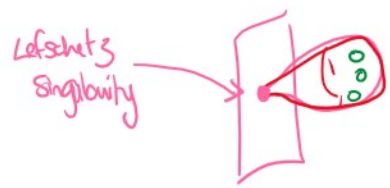


needs too many marked points

Complex deformations	vs.	Graphical homotopies
cx analytic		smooth
immediate changes		can have different intermediate stages

How to get symplectic fillings from graphically homotoped curves?



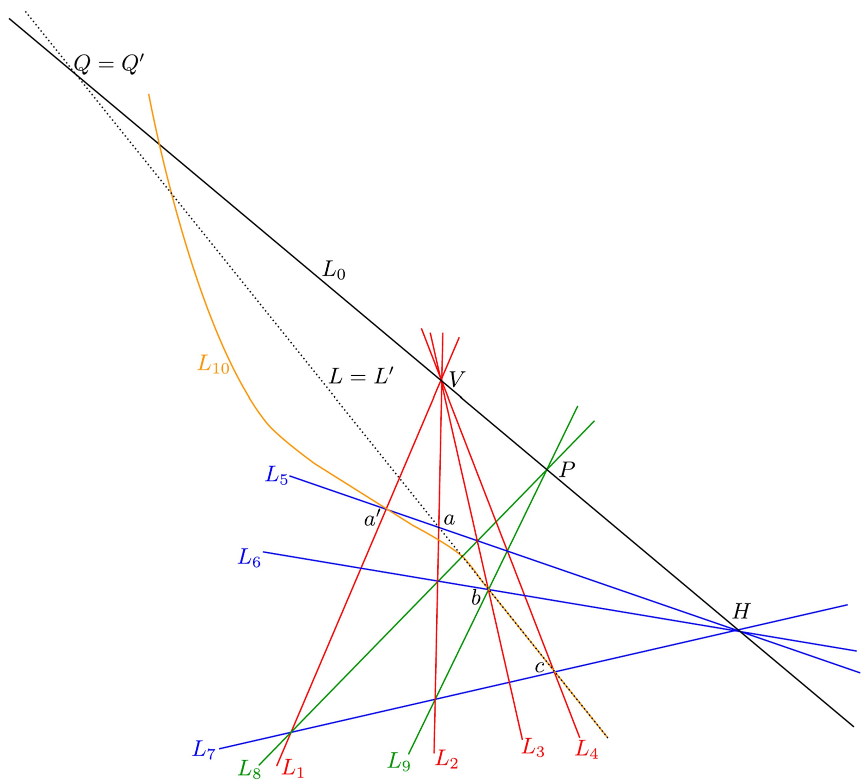


A Lefschetz fibration yields a symplectic filling

Check open book is independent of graphical homotopy

Converse: Wendl ⇝ every sympl. filling has planar Lefschetz fibration

We reverse engineer a curve configuration from the Lefschetz fibration (braided wiring diag)



$Q = Q'$
L_0
L_{10}
$L = L'$
V
P
L_5
a'
a
L_6
b
H
c
L_7
L_8
L_1
L_9
L_2
L_3
L_4